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h–cobordisms between 1–connected 4–manifolds

Matthias Kreck

Mathematisches Institut, Universität Heidelberg
69120 Heidelberg, Federal Republic of Germany

and

Mathematisches Forschungsinstitut Oberwolfach
77709 Oberwolfach, Federal Republic of Germany

Email: `kreck@mathi.uni-heidelberg.de`

Abstract

In this note we classify the diffeomorphism classes rel. boundary of smooth h–cobordisms between two fixed 1–connected 4–manifolds in terms of isometries between the intersection forms.

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In this note we prove the following result.

Theorem *Let M_0 and M_1 be fixed closed oriented smooth 1–connected 4–manifolds. Then the set of diffeomorphism classes rel. boundary of smooth h–cobordisms between M_0 and M_1 is isomorphic to the set of isometries between the intersection forms of M_0 and M_1.*

The same result holds in the topological category if M_0 and M_1 are topological manifolds with same Kirby–Siebenmann invariant k (otherwise there is no h–cobordism between them at all), if we classify up to homeomorphism.

The motivation for our Theorem comes from the fact that the h–cobordism theorem does not hold for smooth h–cobordisms between 4–manifolds [2]. During a discussion with S Donaldson and R Stern about 12 years ago about additional invariants whose vanishing implies that such an h–cobordism is diffeomorphic to the cylinder we wondered how many h–cobordisms exist. The answer above is simpler than in higher dimensions where, due to the existence of exotic spheres, the above Theorem is in general wrong, even if M_0 and M_1 are spheres. The result above implies that a smooth h–cobordism between smooth 1–connected 4–manifolds is the cylinder if and only if there is a diffeomorphism $f\colon M_0 \to M_1$ inducing $(j_*)^{-1}i_*$, where i and j are the inclusions from M_0 and M_1 to W resp. This is of course not the answer one is looking for. A good answer would be that W is a cylinder if and only if the Seiberg–Witten invariants for M_0 and M_1 agree. More precisely the Seiberg Witten invariants (assuming for simplicity $b_2^+(M_i) > 1$) are maps from $\{\alpha \in H^2(M_i) \,|\, \alpha = w_2(M_i) \bmod 2\}$ to the integers. Thus, using the isometry between the intersection forms given by the h–cobordism to identify the cohomology groups, one can compare the Seiberg–Witten invariants of M_0 and M_1. The challenge is to relate the critical values of a Morse function on an h–cobordism to the Seiberg–Witten invariants and to show that the equality of these invariants implies that there is a Morse function without critical values. A relation between the critical values (which is not yet enough to prove the existence of a Morse function without critical values) was recently found by Morgan and Szabo [9] (in the first paragraph of this paper they state that the smooth h–cobordisms are classified by the set of homotopy equivalences, which is not correct, since not every homotopy equivalence between M_0 and M_1 can be realized by an h–cobordism, see below).

The theorem also follows from [7, Proposition 1], where T. Lawson classifies invertible bordisms, and Stalling's result [12] that invertible bordisms and h-cobordisms agree. The proof of Lawson's proposition uses also Stalling's result as well as [11, Proposition 2.1]. The proof of this result is not correct as pointed out and corrected in [1]. Our proof is more direct and elementary.

Proof We will give the proof in the smooth category and discuss the necessary modifications for the topological result at each point.

It is clear that the composition of the inclusion of M_0 into an h–cobordism W between M_0 and M_1 and the homotopy inverse of the inclusion from M_1 is an orientation preserving homotopy equivalence and thus induces an isometry between the intersection forms. This way one obtains a map from the set of diffeomorphism classes rel. boundary of h–cobordisms between M_0 and M_1 to the set of isometries from $H_2(M_0) \to H_2(M_1)$. It is known that this map is surjective. Namely, each isometry can be realized by a homotopy equivalence [8]. And each homotopy equivalence can after composition with a self equivalence of M_1 which operates trivially on $H_2(M_1)$ be realized by a smooth s–cobordism ([13, Theorem 16.5] and the correction in [1] — the proof of this result implies that not every homotopy equivalence can be realized by an h–cobordism). If M_0 and M_1 are topological manifolds with $k(M_0) = k(M_1)$, then it is known that each isometry can be realized by a homeomorphism [3, Theorem 10.1]. This implies surjectivity in the topological case. A different argument for surjectivity both in the smooth and topological category can be found in the proof of [4, Theorem C]. Thus we only have to show injectivity.

Let W and W' be two smooth h–cobordisms between M_0 and M_1 inducing the same isometry between the intersection forms. We will use [6, Theorem 3] to show that W and W' are diffeomorphic rel. boundary. For this we first determine the normal 1–type of an h–cobordism W. By [6, Proposition 2] this is the fibration $B = BSO \to BO$, if $w_2(W) = w_2(M_0) \neq 0$, the non-spin case, and $B = BSpin \to BO$, if $w_2(W) = w_2(M_0) = 0$, the spin case. In the topological case we have to take instead $B = BSTop$ or $B = BSTopSpin$. If we want to apply [6, Theorem 3] we have as a first step to check that normal 1–smoothings of W and W' exist which coincide on the common boundary $M_0 + M_1$. A normal 1–smoothing is in the non-spin case equivalent to an orientation and in the spin case to a spin-structure. Thus, since M_i are simply connected, compatible choices exist.

The next step is to decide if $X = W \cup_{\partial W = \partial W'} W'$ is B–zero-bordant. In the smooth spin case the B–bordism group is spin-bordism which vanishes in dimension 5. In the smooth non-spin case the B–bordism group is oriented bordism which is $\mathbb{Z}/2$ detected by $w_2 \cdot w_3$. One has the same answer in the topological case. One can argue that all 5–manifolds can be made 1–connected by surgery and then they admit a smooth structure since the Kirby–Sibenmann obstruction for the existence of a PL–structure in the 4–th cohomology with $\mathbb{Z}/2$–coefficients vanishes, and in dimension 5 the PL and the smooth categories

are equivalent. In the rest of the argument there is no difference between the smooth and topological case.

Now and later on we need information about the (co)homology of X. For this we choose a fibre homotopy equivalence between X and the mapping torus of the homotopy equivalence on M_0 given by $f = j_0 \cdot (j_1)^{-1} \cdot j_1' \cdot (j_0')^{-1}$, where j_i and j_i' are the inclusions from M_i to W resp. W'. If W and W' induce the same isometry between the intersection forms of M_0 and M_1, then f induces the identity map in second (co)homology. Thus by the Wang sequence for the mapping torus of f we obtain, for arbitrary coefficients, isomorphisms $i^\star\colon H^2(X) \to H^2(M_0)$, where i is the inclusion, and $\delta\colon H^0(M_0) \to H^1(X)$ and $\delta\colon H^2(M_0) \to H^3(X)$.

By the Wu-formulas we have $w_3(X) = Sq^1(w_2(X)) = 0$, since $Sq^1 = 0$ in $H^2(X) \cong H^2(M_0)$. Thus the characteristic number $w_2 \cdot w_3(X)$ vanishes and also in the non-spin case X bounds. Choose in both cases a zero bordism Y and use surgery to make the map $Y \to B$ 3–connected [6, Proposition 4].

The next step is to analyze the surgery obstruction $\theta(Y) \in l_6^{\sim}(1)$. Note that in both cases $\langle w_4(B), \pi_4(B)\rangle \neq 0$ implying that the obstruction is contained in $l_6^{\sim}(1)$ instead of $l_6(1)$ making life easier since we do not have to consider quadratic refinements. The obstruction is given by the equivalence class

$$[H_3(Y,W) \leftarrow \mathrm{im}(d\colon \pi_4(B,Y) \to \pi_3(Y)) \to H_3(Y,W'), \lambda]$$

where the maps are induced by inclusion and λ is the intersection pairing between (Y,W) and (Y,W'). We will show that this obstruction is elementary, ie, there is a submodule $U \subset \mathrm{im}(d\colon \pi_4(B,Y) \to \pi_3(Y))$ such that under both maps U maps to a half rank direct summand and λ vanishes on U. We first note that since $\pi_3(B) = 0$, we can replace $\mathrm{im}(d\colon \pi_4(B,Y) \to \pi_3(Y))$ by $\pi_3(Y)$ and since $\pi_3(Y) \to H_3(Y)$ is surjective we can work with $H_3(Y)$ instead. The situation is here particularly easy since by our homological information both $H_3(Y,W)$ and $H_3(Y,W')$ are isomorphic to $H_3(Y,M_0)$. Thus we have to find $U \subset H_3(Y)$ such that, under inclusion, U maps to a half rank direct summand of $H_3(Y,M_0)$ and λ vanishes on U. Looking at the exact sequence $H_3(Y) \to H_3(Y,M_0) \to H_2(M_0)$ and using that the latter group is free we can pass to rational coefficients. Here we make use of the fact that we do not have to take quadratic refinements into account. Thus the obstruction is elementary if there is $U \subset H_3(Y;\mathbb{Q})$ such that, under inclusion, U maps to a half rank summand of $H_3(Y,M_0;\mathbb{Q})$ and λ vanishes on U. Namely, for such a U choose $U' \subset H_3(Y)$ such that U' is a direct summand in $H_3(Y)$ and $U' \otimes \mathbb{Q} = U$. Since $H_2(M_0)$ is torsion free U' maps to a direct summand in $H_3(Y,M_0)$. If λ vanishes for U the same holds for U' and thus our obstruction is elementary.

Using that $H_4(Y,X;\mathbb{Q}) = H^2(Y;\mathbb{Q}) \cong H^2(B;\mathbb{Q}) = 0$ and $H_2(X,M_0;\mathbb{Q}) = 0$ by the homology information above we obtain an exact sequence

$$0 \to H_3(X,M_0;\mathbb{Q}) \to H_3(Y,M_0;\mathbb{Q}) \to H_3(Y,X;\mathbb{Q}) \to 0.$$

By the homological information above we have isomorphisms

$$H_2(M_0;\mathbb{Q}) \cong H_3(X;\mathbb{Q}) \cong H_3(X,M_0;\mathbb{Q}).$$

Together with the exact sequence

$$0 \to H_3(X;\mathbb{Q}) \to H_3(Y;\mathbb{Q}) \to H_3(Y,X;\mathbb{Q}) \to H_2(X;\mathbb{Q}) = H_2(M_0;\mathbb{Q}) \to 0$$

this implies

$$\mathrm{rank} H_3(Y,M_0;\mathbb{Q}) = 2 \cdot \mathrm{rank} H_2(M_0;\mathbb{Q}) + \mathrm{rank}(\mathrm{coker}(H_3(X;\mathbb{Q}) \to H_3(Y;\mathbb{Q}))).$$

Since the intersection form on $\mathrm{coker}(H_3(X;\mathbb{Q}) \to H_3(Y;\mathbb{Q}))$ is unimodular and skew symmetric there is a submodule $U_1 \subset H_3(Y;\mathbb{Q})$ of half rank of this cokernel, on which the intersction pairing vanishes. Finally the intersection form on the image U_2 of $H_3(X;\mathbb{Q})$ in $H_3(Y;\mathbb{Q})$ is contained in the radical and has rank equal to $\mathrm{rank}(H_2(M_0)$. Thus $U = U_1 \oplus U_2$ is the desired submodule in $H_3(Y;\mathbb{Q})$ implying that the obstruction $\theta(Y)$ is elementary. Then W and W' are diffeomorphic rel. boundary by [6, Theorem 3]. □

I would like to finish the paper with two remarks suggested by the referees. Both concern applications of the theorem above to known results. In the paper [1, Theorem 5.2] the authors show that the map associating to a self equivalence of a smooth (or PL) simply connected closed 4–manifold X the normal invariant is an injection whose image is the kernel of the map into the L–group L_4. We used the latter fact to argue that each self equivalence is induced from an h–cobordism. The injectivity can be derived from the theorem above and the surgery exact sequence.

The other remark concerns pseudo-isotopy classes of closed 1–connected topological 4–manifolds. The theorem above implies that two self homeomorphisms which agree on H_2 are pseudo-isotopic, a result which previously had been proven by Quinn [11] and the author (for diffeomorphisms) [5]. Quinn and independently Perron [10] have shown that pseudo-iosotopy implies isotopy (in the topological category). Thus the group of isotopy classes of homeomorphisms is isomorphic to the isometries of H_2.

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